

# A NOTE ON GEOMETRY OF $\kappa$ -MINKOWSKI SPACE

STEFAN GILLER\*, CEZARY GONERA\*, PIOTR KOSIŃSKI\*, PAWEŁ MAŚLANKA\*

*Department of Theoretical Physics*

*University of Łódź*

*ul. Pomorska 149/153, 90-236 Łódź, Poland*

ABSTRACT. The infinitesimal action of  $\kappa$ -Poincaré group on  $\kappa$ -Minkowski space is computed both for generators of  $\kappa$ -Poincaré algebra and those of Woronowicz generalized Lie algebra. The notion of invariant operators is introduced and generalized Klein-Gordon equation is written out.

## I. INTRODUCTION

In this short note we consider some simple properties of differential operators on  $\kappa$ -Minkowski space  $\mathcal{M}_\kappa$  — a noncommutative deformation of Minkowski space-time which depends on dimensionful parameter  $\kappa$  ([1]). We calculate the infinitesimal action of  $\kappa$ -Poincaré group  $\mathcal{P}_\kappa$  ([1]) on  $\mathcal{M}_\kappa$  both for the generators of  $\kappa$ -Poincaré algebra  $\tilde{\mathcal{P}}_\kappa$  ([2]) (this is done using the duality  $\tilde{\mathcal{P}}_\kappa \Longleftrightarrow \mathcal{P}_\kappa$  described in [3]) and for the elements of Woronowicz generalized Lie algebra ([4]) of  $\kappa$ -Poincaré group ([5]). The result supports the relation between both algebras found in [5]. We introduce also the notion of invariant differential operators on  $\mathcal{M}_\kappa$  and write out the generalized Klein-Gordon equation.

Let us conclude this section by introducing the notions of  $\kappa$ -Poincaré group  $\mathcal{P}_\kappa$  and algebra  $\tilde{\mathcal{P}}_\kappa$ .  $\mathcal{P}_\kappa$  is defined by the following relations ([1])

$$\begin{aligned}
 [x^\mu, x^\nu] &= \frac{i}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu), \\
 [A^\mu{}_\nu, A^\alpha{}_\beta] &= 0, \\
 [A^\mu{}_\nu, x^\rho] &= -\frac{i}{\kappa}((A^\mu{}_0 - \delta_0^\mu)A^\rho{}_\nu + (A^0{}_\nu - \delta_\nu^0)g^{\mu\rho}), \\
 \Delta(A^\mu{}_\nu) &= A^\mu{}_\alpha \otimes A^\alpha{}_\nu, \\
 \Delta(x^\mu) &= A^\mu{}_\alpha \otimes x^\alpha + x^\mu \otimes I, \\
 S(A^\mu{}_\nu) &= A_\nu{}^\mu, \\
 S(x^\mu) &= -A_\nu{}^\mu x^\nu, \\
 \varepsilon(A^\mu{}_\nu) &= \delta_\nu^\mu, \\
 \varepsilon(x^\mu) &= 0.
 \end{aligned} \tag{1}$$

---

\* Supported by Łódź University grant N° 458

The dual structure, the  $\kappa$ -Poincaré algebra  $\widetilde{\mathcal{P}}_\kappa$ , is, in turn, defined as follows ([6])

$$\begin{aligned}
[P_\mu, P_\nu] &= 0, \\
[M_i, M_j] &= i\varepsilon_{ijk}M_k, \\
[M_i, N_j] &= i\varepsilon_{ijk}N_k, \\
[N_i, N_j] &= -i\varepsilon_{ijk}M_k, \\
[M_i, P_0] &= 0, \\
[M_i, P_j] &= i\varepsilon_{ijk}P_k, \\
[N_i, P_0] &= iP_i, \\
[N_i, P_j] &= i\delta_{ij}\left(\frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{i}{\kappa}P_iP_j, \\
\Delta(M_i) &= M_i \otimes I + I \otimes M_i, \\
\Delta(N_i) &= N_i \otimes e^{-P_0/\kappa} + I \otimes N_i - \frac{1}{\kappa}\varepsilon_{ijk}M_j \otimes P_k, \\
\Delta(P_0) &= P_0 \otimes I + I \otimes P_0, \\
\Delta(P_i) &= P_i \otimes e^{-P_0/\kappa} + I \otimes P_i, \\
S(M_i) &= -M_i, \\
S(N_i) &= -N_i + \frac{3i}{2\kappa}P_i, \\
S(P_\mu) &= -P_\mu, \\
\varepsilon(P_\mu, M_i, N_i) &= 0.
\end{aligned} \tag{2}$$

Structures (1), (2) are dual to each other, the duality being fully described in [3].

The analysis given below was suggested to two of the authors (P. Kosiński and P. Maślanka) by J. Lukierski.

## II. $\kappa$ -MINKOWSKI SPACE

The  $\kappa$ -Minkowski space  $\mathcal{M}_\kappa$  ([1]) is a universal  $*$ -algebra with unity generated by four selfadjoint elements  $x^\mu$  subject to the following conditions

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu). \tag{3a}$$

Equipped with the standard coproduct

$$\Delta x^\mu = x^\mu \otimes I + I \otimes x^\mu, \tag{3b}$$

antipode  $S(x^\mu) = -x^\mu$  and counit  $\varepsilon(x^\mu) = 0$  it becomes a quantum group.

On  $\mathcal{M}_\kappa$  one can construct a bicovariant five-dimensional calculus which is defined

by the following relations ([5])

$$\begin{aligned}
\tau^\mu &\equiv dx^\mu, & \tau &\equiv d\left(x^2 + \frac{3i}{\kappa}x^0\right) - 2x_\mu dx^\mu, \\
[\tau^\mu, x^\nu] &= \frac{i}{\kappa}g^{0\mu}\tau^\nu - \frac{i}{\kappa}g^{\mu\nu}\tau^0 + \frac{1}{4}g^{\mu\nu}\tau, \\
[\tau, x^\mu] &= -\frac{4}{\kappa^2}\tau^\mu, \\
\tau^\mu \wedge \tau^\nu &= -\tau^\nu \wedge \tau^\mu, \\
\tau \wedge \tau^\mu &= -\tau^\mu \wedge \tau, \\
(\tau^\mu)^* &= \tau^\mu, & \tau^* &= -\tau, \\
d\tau^\mu &= 0, \\
d\tau &= -2\tau_\mu \wedge \tau^\mu.
\end{aligned} \tag{4}$$

The  $\kappa$ -Minkowski space carries a left-covariant action of  $\kappa$ -Poincaré group  $\mathcal{P}_\kappa$  ([1]),  $\rho_L : \mathcal{M}_\kappa \rightarrow \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$ , given by

$$\rho_L(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes I. \tag{5}$$

The calculus defined by (4) is covariant under the action of  $\mathcal{P}_\kappa$  which reads

$$\begin{aligned}
\tilde{\rho}_L(\tau^\mu) &= \Lambda^\mu{}_\nu \otimes \tau^\nu, \\
\tilde{\rho}_L(\tau) &= I \otimes \tau.
\end{aligned} \tag{6}$$

### III. DERIVATIVES, INFINITESIMAL ACTIONS AND INVARIANT OPERATORS

The product of generators  $x^\mu$  will be called normally ordered if all  $x^0$  factors stand leftmost. This definition can be used to ascribe a unique element  $:f(x):$  of  $\mathcal{M}_\kappa$  to any polynomial function of four variables  $f$ . Formally, it can be extended to any analytic function  $f$ .

Let us now one define the (left) partial derivatives: for any  $f \in \mathcal{M}_\kappa$  we write

$$df = \partial_\mu f \tau^\mu + \partial f \tau. \tag{7}$$

It is a matter of some boring calculations (using the commutation rules (3a)) to find the following formula

$$\begin{aligned}
d : f : &:= \left( \kappa \sin\left(\frac{\partial_0}{\kappa}\right) + \frac{i}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f : \tau^0 + : e^{i\frac{\partial_0}{\kappa}} \frac{\partial f}{\partial x^i} : \tau^i \\
&+ : \left( \frac{\kappa^2}{4} \left( 1 - \cos\left(\frac{\partial_0}{\kappa}\right) \right) - \frac{1}{8} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f : \tau
\end{aligned} \tag{8}$$

or

$$\begin{aligned}
\partial_0 : f : &:= \left( \kappa \sin\left(\frac{\partial_0}{\kappa}\right) + \frac{i}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f : \\
\partial_i : f : &:= e^{i\frac{\partial_0}{\kappa}} \frac{\partial f}{\partial x^i} : \\
\partial : f : &:= \left( \frac{\kappa^2}{4} \left( 1 - \cos\left(\frac{\partial_0}{\kappa}\right) \right) - \frac{1}{8} e^{i\frac{\partial_0}{\kappa}} \Delta \right) f :
\end{aligned} \tag{9}$$

Let us now define the infinitesimal action of  $\mathcal{P}_\kappa$  on  $\mathcal{M}_\kappa$ . Let  $X$  be any element of the Hopf algebra dual to  $\mathcal{P}_\kappa$  — the  $\kappa$ -Poincaré algebra  $\tilde{\mathcal{P}}_\kappa$  (cf. [3] for the proof of duality). The corresponding infinitesimal action

$$\hat{X} : \mathcal{M}_\kappa \rightarrow \mathcal{M}_\kappa$$

is defined as follows: for any  $f \in \mathcal{M}_\kappa$ ,

$$\hat{X}f = (X \otimes \text{id}) \circ \rho_L(f). \quad (10)$$

Using the standard duality rules ([3]), we conclude that

$$\begin{aligned} \hat{P}_\mu x^\alpha &= i\delta_\mu^\alpha, \\ \hat{P}_\mu : x^\alpha x^\beta &:= i\delta_\mu^\beta x^\alpha + i\delta_\mu^\alpha x^\beta \end{aligned} \quad (11)$$

etc. One can show that, in general,

$$\hat{P}_\mu : f := i \frac{\partial f}{\partial x^\mu} : \quad (12)$$

Also, using the fact that  $\tilde{\rho}_L$  is a left action of  $\mathcal{P}_\kappa$  on  $\mathcal{M}_\kappa$  together with the duality  $\mathcal{P}_\kappa \rightarrow \tilde{\mathcal{P}}_\kappa$ , we conclude that

$$F(\hat{P}_\mu) : f := F\left(i \frac{\partial f}{\partial x^\mu}\right) f : \quad (13)$$

Formulae (11)–(13) have the following interpretation. In [5] the fifteen-dimensional bicovariant calculus on  $\mathcal{P}_\kappa$  has been constructed using the methods developed by Woronowicz ([4]). The resulting generalized Lie algebra is also fifteen-dimensional, the additional generators being the generalized mass square operator and the components of generalized Pauli-Lubanski fourvector. All generators of this Lie algebra can be expressed in terms of the generators  $P_\mu$ ,  $M_{\alpha\beta}$  of  $\tilde{\mathcal{P}}_\kappa$  ([5]). In particular, the translation generators  $\chi_\mu$  as well as the mass squared operator  $\chi$  are expressible in terms of  $P_\mu$  only. The relevant expressions are given by formulae (20) of [5]. Comparing them with (9), (13) above, we conclude that

$$\begin{aligned} \hat{\chi}_\mu &\equiv \partial_\mu, \\ \hat{\chi} &\equiv \partial. \end{aligned} \quad (14)$$

These relations, obtained here by explicit computations, follow also from (7) if one takes into account that  $\mathcal{M}_\kappa$  is a quantum subgroup of  $\mathcal{P}_\kappa$ .

It is also not difficult to obtain the action of Lorentz generators. Combining (1) and (3a) with the duality  $\mathcal{P}_\kappa \rightarrow \tilde{\mathcal{P}}_\kappa$  described in detail in [5], we conclude first that the action of  $M_i$  and  $N_i$  coincides with the proposal of Majid and Ruegg ([6]); the actual computation is then easy and gives

$$\begin{aligned} \hat{M}_i : f(x^\mu) &:= -i\varepsilon_{ijl}x^j \frac{\partial f(x^\mu)}{\partial x^l} : \\ \hat{N}_i : f(x^\mu) &:= \left( ix^0 \frac{\partial}{\partial x^i} + x^i \left( \frac{\kappa}{2} \left( 1 - e^{\frac{-2i}{\kappa} \frac{\partial}{\partial x^0}} \right) - \frac{1}{2\kappa} \Delta \right) \right. \\ &\quad \left. + \frac{1}{\kappa} x^k \frac{\partial^2}{\partial x^k \partial x^i} \right) f(x^\mu) : \end{aligned} \quad (15)$$

Let us now pass to the notion of invariant operator;  $\widehat{C}$  is an invariant operator on  $\mathcal{M}_\kappa$  if

$$\rho_L \circ \widehat{C} = (\text{id} \otimes \widehat{C}) \circ \rho_L. \quad (16)$$

We shall show that if  $C$  is a central element of  $\widetilde{\mathcal{P}}_\kappa$ , then

$$\widehat{C}f = (C \otimes \text{id}) \circ \rho_L(f) \quad (17)$$

is an invariant operator. To prove this let us take any  $Y \in \widetilde{\mathcal{P}}_\kappa$ , then

$$YC = CY \quad (18)$$

or, in other words, for any  $a \in \mathcal{P}_\kappa$ ,

$$Y(a_{(1)})C(a_{(2)}) = C(a_{(1)})Y(a_{(2)}) \quad (19)$$

where  $\Delta a = a_{(1)} \otimes a_{(2)}$ . Let us fix  $a$  and write (19) as

$$Y(a_{(1)}C(a_{(2)})) = Y(a_{(2)}C(a_{(1)})). \quad (20)$$

As (20) holds for any  $Y \in \widetilde{\mathcal{P}}_\kappa$  we conclude that for any  $a \in \mathcal{P}_\kappa$

$$C(a_{(2)})a_{(1)} = C(a_{(1)})a_{(2)}. \quad (21)$$

Now let

$$\begin{aligned} \rho_L(x) &= a_{(1)} \otimes x_{(1)}, \\ \rho_L(x_{(1)}) &= a_{(1)(2)} \otimes x_{(2)}, \\ \Delta a_{(1)} &= a_{(1)}^{(1)} \otimes a_{(1)}^{(2)}. \end{aligned} \quad (22)$$

The identity

$$(\text{id} \otimes \rho_L) \circ \rho_L = (\Delta \otimes \text{id}) \circ \rho_L \quad (23)$$

implies

$$a_{(1)} \otimes a_{(1)(2)} \otimes x_{(2)} = a_{(1)}^{(1)} \otimes a_{(1)}^{(2)} \otimes x_{(1)}. \quad (24)$$

Applying to both sides  $\text{id} \otimes C \otimes \text{id}$  and  $C \otimes \text{id} \otimes \text{id}$ , we get

$$\begin{aligned} C(a_{(1)(2)})a_{(1)} \otimes x_{(2)} &= C(a_{(1)}^{(2)})a_{(1)}^{(1)} \otimes x_{(1)}, \\ C(a_{(1)})a_{(1)(2)} \otimes x_{(2)} &= C(a_{(1)}^{(1)})a_{(1)}^{(2)} \otimes x_{(1)}. \end{aligned} \quad (25)$$

It follows from (21) applied to  $a_{(1)}$  that the right-hand sides of (25) are equal. So,

$$C(a_{(1)(2)})a_{(1)} \otimes x_{(2)} = C(a_{(1)})a_{(1)(2)} \otimes x_{(2)} \quad (26)$$

i.e.

$$(\text{id} \otimes \widehat{C}) \circ \rho_L(x) = \rho_L \circ \widehat{C}(x). \quad (27)$$

Using the above result we can easily construct the deformed Klein-Gordon equation. Namely, we take as a central element the counterpart of mass squared Casimir operator  $\chi$  ([5]). Due to (14) the generalized Klein-Gordon equation reads

$$\left(\partial + \frac{m^2}{8}\right)f = 0; \quad (28)$$

the coefficient  $\frac{1}{8}$  is dictated by the correspondence with standard Klein-Gordon equation in the limit  $\kappa \rightarrow \infty$ . Let us note that (28) can be written, due to (9), in the form

$$\left[\partial_0^2 - \partial_i^2 + m^2 \left(1 + \frac{m^2}{4\kappa^2}\right)\right]f = 0; \quad (29)$$

here  $\partial_0, \partial_i$  are the operators given by (9). It seems therefore that the Woronowicz operators  $\chi_\mu$  are better candidates for translation generators than  $P_\mu$ 's. Note that the operators  $\chi_\mu$  already appeared in [7], [8].

## REFERENCES

- [1] S. Zakrzewski, J. Phys. **A 27** (1994), 2075.
- [2] J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. **B 293** (1992), 344.
- [3] P. Kosiński, P. Maślanka, *The duality between  $\kappa$ -Poincaré algebra and  $\kappa$ -Poincaré group*, hep-th.
- [4] S.L. Woronowicz, Comm. Math. Phys. **122** (1989), 125.
- [5] P. Kosiński, P. Maślanka, J. Sobczyk, in: *Proc. of IV Coll. on Quantum Groups and Integrable Sysytems, Prague 1995, to be published*.
- [6] S. Majid, H. Ruegg, Phys. Lett. **B 334** (1994), 348.
- [7] H. Ruegg, V. Tolstoy, Lett. Math. Phys. **32** (1994), 85.
- [8] J. Lukierski, H. Ruegg, V. Tolstoy, in: *Quantum Groups. Formalism and Applications. Proc. of XXX Karpacz Winter School of Theoretical Physics*, ed. J. Lukierski, Z. Popowicz, J. Sobczyk, PWN 1995.